

# A decomposition theorem for fuzzy set-valued random variables and a characterization of fuzzy random translation

Giacomo Aletti<sup>\*1,2</sup> and Enea G. Bongiorno<sup>†1</sup>

<sup>1</sup>Dipartimento di Matematica, Università degli studi di Milano

<sup>2</sup>ADAMSS CENTRE (ADvanced Applied Mathematical and Statistical Sciences)

November 28, 2011

## Abstract

Let  $X$  be a fuzzy set-valued random variable (FRV), and  $\Theta_X$  the family of all fuzzy sets  $B$  for which the Hukuhara difference  $X \ominus_H B$  exists  $\mathbb{P}$ -almost surely. In this paper, we prove that  $X$  can be decomposed as  $X(\omega) = C \oplus Y(\omega)$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ ,  $C$  is the unique deterministic fuzzy set that minimizes  $\mathbb{E}[d_2(X, B)^2]$  as  $B$  is varying in  $\Theta_X$ , and  $Y$  is a centered FRV (i.e. its generalized Steiner point is the origin). This decomposition allows us to characterize all FRV translation (i.e.  $X(\omega) = M \oplus \mathbb{I}_{\xi(\omega)}$  for some deterministic fuzzy convex set  $M$  and some random element in  $\mathbb{R}^d$ ). In particular,  $X$  is a FRV translation if and only if the Aumann expectation  $\mathbb{E}X$  is equal to  $C$  up to a translation. Examples, such as the Gaussian case, are provided.

**Keywords:** Fuzzy random variable; fuzzy random translation; Gaussian fuzzy random set; Aumann expectation; Hukuhara difference; decomposition theorem; randomness defuzzification;

## Introduction

It is widely known (e.g. [5, Theorem 6.1.7]) that a Gaussian fuzzy random variable may be decomposed as

$$X = \mathbb{E}X \oplus \mathbb{I}_{\xi}, \quad (1)$$

where  $\mathbb{E}X$  is the expectation of  $X$  in the Aumann sense,  $\xi$  is a Gaussian random element in  $\mathbb{R}^d$  with  $\mathbb{E}\xi = 0$  and  $\mathbb{I}_A : \mathbb{R}^d \rightarrow \{0, 1\}$  denotes the indicator function of any  $A \subseteq \mathbb{R}^d$

$$\mathbb{I}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

We write  $\mathbb{I}_a$  instead of  $\mathbb{I}_{\{a\}}$  whenever  $A = \{a\}$  is a singleton. Roughly speaking, a Gaussian FRV  $X$  is just a deterministic fuzzy set (its expected value  $\mathbb{E}X$ ) up to a Gaussian translation  $\xi$  which carries out all the randomness of  $X$ . In this view, Equation (1) entails a *randomness defuzzification* for the Gaussian FRV  $X$  according to which the underlying probability structure can be defined just only on  $\mathbb{R}^d$  and no longer on  $\mathbb{F}$ , the space of normal fuzzy sets with compact convex level sets. Such randomness defuzzification occurs whenever a FRV  $X$  is a random translation of a deterministic fuzzy set  $M$ . In this paper we provide a characterization for random translations by means of a suitable decomposition theorem that holds for any FRV. In particular, given a centered FRV  $X$ , we define the family  $\Theta_X$  of all deterministic  $B \in \mathbb{F}$  for which the Hukuhara difference  $X \ominus_H B$  exists almost surely. We show that this

---

<sup>\*</sup>giacomo.aletti@unimi.it

<sup>†</sup>enea.bongiorno@unimi.it

set is not empty, convex and closed in  $(\mathbb{F}, d_2)$ , where  $d_2$  corresponds to the  $L^2$  metric in the space of support functions. Further,

$$C = \arg \min_{U \in \Theta_X} \mathbb{E}(d_2(X, U)^2)$$

is unique and there exists a FRV  $Y$  such that  $X(\omega) = C \oplus Y(\omega)$ ; in some sense,  $C$  and  $Y$  are the deterministic part (with respect to  $\oplus$ ) and the random part of  $X$  respectively. Since, the Aumann expectation  $\mathbb{E}X$  is the (unique) Fréchet expectation with respect to  $d_2$ , i.e.

$$\mathbb{E}X = \arg \min_{U \in \mathbb{F}} \mathbb{E}(d_2(X, U)^2),$$

we obtain immediately that a FRV  $X$  is a random translation of  $C$  (i.e.  $Y(\omega)$  is almost surely a singleton) if and only if  $\mathbb{E}X$  is equal to  $C$ .

The paper is organized as follow. Section 1 introduces necessary notations and literature results. Section 2 studies properties of the Hukuhara set  $\Theta_X$  whilst Section 3 presents the decomposition theorem of FRV and the characterization of FRV translation.

## 1 Preliminaries

Denote by  $\mathbb{K}$  the class of non-empty compact convex subsets of  $\mathbb{R}^d$ , endowed with the Hausdorff metric

$$\delta_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

and the operations

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda \cdot A = \lambda A = \{\lambda a : a \in A\} \quad \text{with } \lambda > 0.$$

For a non-empty closed convex set  $A \subset \mathbb{R}^d$  the *support function*  $s_A : S^{d-1} \rightarrow \mathbb{R}$  is defined by

$$s_A(x) = \sup\{\langle x, a \rangle : a \in A\}, \quad \text{for } x \in S^{d-1},$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^d$  and  $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$  is the unit sphere in  $\mathbb{R}^d$ . The *Steiner point* of  $A \in \mathbb{K}$  is defined by

$$\mathbf{ste}(A) = \frac{1}{v_d} \int_{S^{d-1}} s_A(x) x \, d\lambda(x)$$

where  $x \in S^{d-1}$  varies over the unit vectors of  $\mathbb{R}^d$ ,  $\lambda$  is the Lebesgue measure on  $S^{d-1}$ , and  $v_d$  is the volume of the unit ball of  $\mathbb{R}^d$ .

**Fuzzy Sets.** A *fuzzy set* is a map  $\nu : \mathbb{R}^d \rightarrow [0, 1]$ . Let  $\mathbb{F}$  denote the family of all fuzzy sets  $\nu$ , which satisfy the following conditions.

1.  $\nu$  is an upper semicontinuous function, i.e. for each  $\alpha \in (0, 1]$ , the *cut set* or the  $\alpha$ -*level set*  $\nu_\alpha = \{x \in \mathbb{R}^d : \nu(x) \geq \alpha\}$  is a closed subset of  $\mathbb{R}^d$ .
2.  $\nu$  is normal; i.e.  $\nu_1 = \{x \in \mathbb{R}^d : \nu(x) = 1\} \neq \emptyset$ .
3. The support set  $\nu_0 = \overline{\{x \in \mathbb{R}^d : \nu(x) > 0\}}$  of  $\nu$  is compact; hence every  $\nu_\alpha$  is compact for  $\alpha \in (0, 1]$ .
4. For any  $\alpha \in [0, 1]$ ,  $\nu_\alpha$  is a convex subset of  $\mathbb{R}^d$ .

For any  $\nu \in \mathbb{F}$  define the *support function* of  $\nu$  as follows:

$$s_\nu(x, \alpha) = \begin{cases} s_{\nu_\alpha}(x) & \text{if } \alpha > 0, \\ s_{\nu_0}(x) & \text{if } \alpha = 0, \end{cases}$$

for  $(x, \alpha) \in S^{d-1} \times [0, 1]$ . Let us endow  $\mathbb{F}$  with the operations

$$(\nu^1 \oplus \nu^2)_\alpha = \nu_\alpha^1 + \nu_\alpha^2, \quad (\lambda \odot \nu^1)_\alpha = \lambda \cdot \nu_\alpha^1, \quad \text{with } \lambda > 0$$

(so that  $(\mathbb{F}, \oplus, \cdot)$  is a convex cone), and with the metrics

$$\delta_H^\infty(\nu^1, \nu^2) = \sup\{\alpha \in [0, 1] : \delta_H(\nu_\alpha^1, \nu_\alpha^2)\},$$

$$d_2(\nu^1, \nu^2) = \left( \int_0^1 \int_{S^{d-1}} |s_{\nu^1}(\alpha, u) - s_{\nu^2}(\alpha, u)|^2 \, du \, d\alpha \right)^{\frac{1}{2}}.$$

It is known that  $(\mathbb{F}, \delta_H^\infty)$  is a complete metric space while  $(\mathbb{F}, d_2)$  is not (cf. [4, Chapter 7]). The *generalized Steiner point* of  $A \in \mathbb{F}$  is defined by

$$\mathbf{Ste}(A) = \int_{[0,1]} \mathbf{ste}(A_\alpha) \, d\alpha,$$

where  $d\alpha$  is the Lebesgue measure on  $[0, 1]$ . In other words,  $\mathbf{Ste}(A)$  may be seen as a weighted average of steiner points of the level sets of  $A$ . The following properties are satisfied (cf. [10]).

1. For any  $A \in \mathbb{F}$ ,  $\mathbf{Ste}(A) \in A_0$ .
2. For any  $A, B \in \mathbb{F}$ ,  $\mathbf{Ste}(A \oplus B) = \mathbf{Ste}(A) + \mathbf{Ste}(B)$ .
3.  $\mathbf{Ste} : \mathbb{F} \rightarrow \mathbb{R}^d$  is continuous.

**On the support function for fuzzy sets.** It is known that the support function for a fuzzy set  $\nu \in \mathbb{F}$  can be defined equivalently on the closed unit ball  $B(0, 1) = \{x \in \mathbb{R}^d : \|x\| \leq 1\} \subset \mathbb{R}^d$  instead of the unit sphere  $S^{d-1}$  by

$$\begin{aligned} s_\nu^* : B(0, 1) &\rightarrow \mathbb{R} \\ x &\mapsto s_\nu^*(x) = \max\{\langle x, y \rangle : y \in \mathbb{R}^d, \nu(y) \geq \|x\|\}. \end{aligned}$$

In particular, the following relationship between support function definitions hold

$$\begin{aligned} \forall (x, \alpha) \in S^{d-1} \times [0, 1], \quad s_\nu(x, \alpha) &= \begin{cases} s_\nu^*(\alpha x), & \text{if } \alpha \neq 0; \\ \sup_{y \in \nu_0} \langle y, x \rangle, & \text{if } \alpha = 0. \end{cases} \\ \forall x \in B(0, 1), \quad s_\nu^*(x) &= \begin{cases} \|x\| s_\nu\left(\frac{x}{\|x\|}, \|x\|\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \end{aligned}$$

In [1], the author prove that a function  $f : B(0, 1) \rightarrow \mathbb{R}$  is a support function of some fuzzy set  $\nu \in \mathbb{F}$  if and only if the following six properties are satisfied:

(Property.1)  $f$  is upper semicontinuous, i.e.,

$$f(x) = \limsup_{y \rightarrow x} f(y), \quad \forall x \in B(0, 1).$$

(Property.2)  $f$  is positively semihomogeneous, i.e.,

$$\lambda f(x) \leq f(\lambda x), \quad \forall \lambda \in (0, 1], \forall x \in B(0, 1).$$

(Property.3)  $f$  is quasiadditive, i.e.,

$$\|x\| f\left(\lambda \frac{x}{\|x\|}\right) \leq \|x_1\| f\left(\lambda \frac{x_1}{\|x_1\|}\right) + \|x_2\| f\left(\lambda \frac{x_2}{\|x_2\|}\right),$$

for every  $\lambda \in (0, 1]$ , and  $x, x_1, x_2 \in \mathbb{R}^d \setminus \{0\}$ , with  $x = x_1 + x_2$ .

(Property.4)  $f$  is normal, i.e.,

$$f(x) + f(-x) \geq 0, \quad \forall x \in B(0, 1).$$

(Property.5)  $f(\cdot)/\|\cdot\|$  is bounded, i.e.,

$$\sup\{f(x)/\|x\| : x \in B(0, 1) \setminus \{0\}\} < \infty.$$

(Property.6)  $f(0) = 0$ .

**Embeddings.** Let  $C(S^{d-1})$  denote the Banach space of all continuous functions  $v$  on  $S^{d-1}$  with respect to the norm  $\|v\|_C = \sup_{x \in S^{d-1}} |v(x)|$ . Let  $\overline{\mathcal{C}} := \overline{\mathcal{C}}([0, 1], C(S^{d-1}))$  be the set of all functions  $f : [0, 1] \rightarrow C(S^{d-1})$  such that  $f$  is bounded, left continuous with respect to  $\alpha \in (0, 1]$ , right continuous at 0, and  $f$  has right limit for any  $\alpha \in (0, 1)$ . Then we have that  $\overline{\mathcal{C}}$  is a Banach space with the norm  $\|f\|_{\overline{\mathcal{C}}} = \sup_{\alpha \in [0, 1]} \|f(\alpha)\|_C$ .

Let  $\mathcal{L} := L^2([0, 1] \times S^{d-1}; \mathbb{R})$  be the Hilbert space of square integrable real-valued functions defined on  $[0, 1] \times S^{d-1}$ .

It is known, cf. [5, 8, 9], that the injection  $j$  defined by

$$\begin{aligned} j : \mathbb{F} &\rightarrow \overline{\mathcal{C}} \cap \mathcal{L} \\ \nu &\mapsto j(\nu) = s_\nu, \end{aligned} \tag{2}$$

satisfies the following properties:

1.  $j(r\nu^1 \oplus t\nu^2) = rj(\nu^1) + tj(\nu^2)$ ,  $\nu^1, \nu^2 \in \mathbb{F}$  and  $r, t \geq 0$ .
2.  $j$  is an isometric mapping, i.e. for every  $\nu^1, \nu^2 \in \mathbb{F}$ ,

$$\delta_H^\infty(\nu^1, \nu^2) = \|j(\nu^1) - j(\nu^2)\|_{\overline{\mathcal{C}}}, \quad \text{and} \quad d_2(\nu^1, \nu^2) = \|j(\nu^1) - j(\nu^2)\|_{\mathcal{L}}.$$

**Fuzzy random variables.** Let  $(\Omega, \mathfrak{F}, \mathbb{P})$  be a complete probability space. A *fuzzy set-valued random variable* (FRV) is a function  $X : \Omega \rightarrow \mathbb{F}$ , such that  $X_\alpha : \omega \mapsto X(\omega)_\alpha$  are random compact convex sets for every  $\alpha \in (0, 1]$  (i.e.  $X_\alpha$  is a  $\mathbb{K}$ -valued function measurable w.r.t.  $\mathcal{B}_{\mathbb{K}}$ , the Borel  $\sigma$ -algebra on  $\mathbb{K}$  generated by the metric  $\delta_H$ ). It has been proven in [3] that this measurability definition is equivalent to the  $\mathcal{B}(\mathbb{F}, d_2)$ -measurability and, it is necessary (but not sufficient) for the  $\mathcal{B}(\mathbb{F}, \delta_H^\infty)$ -measurability, where  $\mathcal{B}(\mathbb{F}, D)$  denotes the Borel  $\sigma$ -algebra defined on  $\mathbb{F}$  w.r.t. the metric  $D$ .

As a consequence of continuity of  $\mathbf{Ste}(\cdot)$ , if  $X$  is a FRV, then  $\mathbf{Ste}(X)$  is a random element in  $\mathbb{R}^d$ . A FRV  $X$  is *integrably bounded* and we write  $X \in L^1[\Omega; \mathbb{F}]$ , if  $\mathbb{E}(\sup_{x \in X_0} \|x\|) < +\infty$ . The (Aumann) *expected value* of  $X \in L^1[\Omega; \mathbb{F}]$ , denoted by  $\mathbb{E}[X]$ , is a fuzzy set such that, for every  $\alpha \in [0, 1]$ ,

$$(\mathbb{E}[X])_\alpha = \int_{\Omega} X_\alpha \, d\mathbb{P} = \{\mathbb{E}(f) : f \in L^1[\Omega; \mathbb{R}^d], f \in X_\alpha \, \mathbb{P} - \text{a.e.}\}.$$

It should be pointed out that, whenever  $\mathbb{E}[(\sup_{x \in X_0} \|x\|)^2] < +\infty$  (we write  $X \in L^2[\Omega; \mathbb{F}]$ ), the expected value in the Aumann's sense is even the Fréchet expectation with respect to  $d_2$ , i.e.

$$\mathbb{E}X = \arg \min_{U \in \mathbb{F}} \mathbb{E}(d_2(X, U)^2),$$

see for example [7].

In view of above measurability consideration and from embedding (2) it follows that every FRV  $X$  can be regarded as a random element in  $\mathcal{L}$ , where  $s_X(\cdot, \cdot)(\omega) = s_{X(\omega)}(\cdot, \cdot)$ . Moreover, if  $X \in L^1[\Omega; \mathbb{F}]$ , for any  $(x, \alpha) \in \mathbb{R}^d \times [0, 1]$ ,  $s_{X(\cdot)}(x, \alpha) \in L^1[\Omega; \mathbb{R}]$  and

$$\mathbb{E}[s_X(x, \alpha)] = s_{\mathbb{E}X}(x, \alpha). \tag{3}$$

Finally, let  $L^2[\Omega; \mathcal{L}] := \{f : \Omega \rightarrow \mathcal{L} \text{ s.t. } [\int_{\Omega} \|f(\omega)\|_{\mathcal{L}}^2 \, d\mathbb{P}]^{1/2} < +\infty\}$ . It is easy to show that the map

$$\begin{aligned} J : L^2[\Omega; \mathbb{F}] &\rightarrow L^2[\Omega; \mathcal{L}] \\ X &\mapsto J(X) = j(X(\cdot)) = s_{X(\cdot)}, \end{aligned}$$

is well-defined and induces an isometry in the following sense: for every  $X^1, X^2 \in L^2[\Omega; \mathbb{F}]$ ,

$$\Delta_2(X^1, X^2) := \mathbb{E}(d_2(X^1, X^2)) = \mathbb{E}(\|J(X^1) - J(X^2)\|_{\mathcal{L}}).$$

## 2 Hukuhara set

In this section we define the *Hukuhara set* associated to a FRV  $X$ , namely  $\Theta_X$ . We provide some properties of  $\Theta_X$  most of which turn out to be useful in the next section where a decomposition theorem for fuzzy random variables is set.

Let  $K$  be in  $\mathbb{F}$  such that  $\mathbf{Ste}(K) = 0$  and consider

$$\theta_K = \{B \in \mathbb{F} : \mathbf{Ste}(B) = 0 \text{ and } \exists A \in \mathbb{F} \text{ s.t. } B \oplus A = K\};$$

i.e. the family of those centered convex compact fuzzy sets  $B$  for which the Hukuhara difference  $K \ominus_H B$  does exist. Note that  $\theta_K$  is not empty, since  $\mathbb{I}_0, K \in \theta_K$  and  $\{\lambda \odot K\}_{\lambda \in [0,1]} \subseteq \theta_K$ . Clearly, if  $B \in \theta_K$  and  $A$  is the Hukuhara difference between  $K$  and  $B$ , then  $A \in \theta_K$ .

**Proposition 1**  $\theta_K$  is a closed subset in  $(\mathbb{F}, \delta_H^\infty)$ .

**Proof.** Let  $\{B_n\} \subset \theta_K$  be a convergent sequence with limit  $B \in \mathbb{F}$  with respect to  $\delta_H^\infty$ , we have to prove that  $B \in \theta_K$ . Equivalently, we have to prove that there exists  $A \in \mathbb{F}$  such that  $B \oplus A = X$ . For each  $n = 1, 2, \dots$  there exist  $A_n \in \mathbb{F}$  such that  $B_n \oplus A_n = K$ . Thus, the idea is to prove that  $\{A_n\}_{n=1}^\infty$  converges, w.r.t.  $\delta_H^\infty$ , to some  $A \in \mathbb{F}$  such that  $B \oplus A = X$ . To do this, let us consider the following chains of equalities

$$\begin{aligned} \delta_H^\infty(A_m, A_n) &= \|s_{A_m} - s_{A_n}\|_{\overline{C}} \\ &= \|(s_{A_m} + s_{B_m}) - (s_{A_n} + s_{B_n}) + s_{B_n} - s_{B_m}\|_{\overline{C}} \\ &= \|s_K - s_K + s_{B_n} - s_{B_m}\|_{\overline{C}} \\ &= \|s_{B_n} - s_{B_m}\|_{\overline{C}} = \delta_H^\infty(B_n, B_m) \rightarrow 0, \quad \text{for } n, m \rightarrow \infty \end{aligned}$$

where we use the isometry  $A \mapsto s_A$  (first and last equalities) and the fact that  $B_n, B_m$  belong to  $\theta_K$  (third equality). Above limit implies that  $\{A_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathbb{F}, \delta_H^\infty)$  that is a complete metric space (e.g. [5, Theorem 5.1.6]), and then there exists  $A$  in  $\mathbb{F}$  such that  $A_n \rightarrow A$ . As a consequence,  $B_n \oplus A_n \rightarrow B \oplus A$  for  $n \rightarrow \infty$  combined with

$$0 = \delta_H^\infty(B_n \oplus A_n, X),$$

guarantees that  $B \oplus A = X$  and hence  $B \in \theta_K$ ; that is the thesis.  $\blacksquare$

In what follows we need the next lemma according to which a fuzzy set can be defined starting from its  $\alpha$ -cuts.

**Lemma 2** (See [4, Proposition 6.1.7, p.39]) *If  $\{C_\alpha\}_{\alpha \in [0,1]}$  satisfies*

- (a)  $C_\alpha$  is a non empty compact convex subset of  $\mathbb{R}^d$ , for every  $\alpha \in [0, 1]$ ;
- (b)  $C_\beta \subseteq C_\alpha$  for  $0 \leq \alpha \leq \beta \leq 1$ ;
- (c)  $C_\alpha = \bigcap_{i=1}^\infty C_{\alpha_i}$  for all sequence  $\{\alpha_i\}_{i \in \mathbb{N}}$  in  $[0, 1]$  converging from below to  $\alpha$ , i.e.  $\alpha_i \uparrow \alpha$  in  $[0, 1]$ ;

then the function

$$\nu(x) = \begin{cases} 0, & \text{if } x \notin C_0, \\ \sup\{\alpha \in [0, 1] : x \in C_\alpha\}, & \text{if } x \in C_0, \end{cases}$$

is an element of  $\mathbb{F}$  with  $\nu_\alpha = C_\alpha$  for any  $\alpha \in (0, 1]$  and

$$\nu_0 = \overline{\bigcup_{\alpha \in (0,1]} C_\alpha} \subseteq C_0.$$

Let  $X$  be a FRV. For the sake of simplicity and without loss of generality, let us suppose that  $\mathbf{Ste}(X) = 0$ ; otherwise one can always considered its associated centered FRV  $\tilde{X} = X - \mathbb{I}_{\mathbf{Ste}(X)}$ . Next theorem defines the *Hukuhara set*  $\Theta_X$  associated to  $X$ , and provides some properties of  $\Theta_X$ .

**Proposition 3** *If  $B \in \mathbb{F}$ , then  $E = \{B \in \theta_X\} := \{\omega \in \Omega : B \in \theta_{X(\omega)}\}$  is measurable in  $(\Omega, \mathfrak{F})$ . Moreover, if  $\Theta_X = \{B \in \mathbb{F} : \mathbb{P}(B \in \theta_X) = 1\}$ , then the following statements hold.*

- (i)  $\Theta_X$  is non-empty.
- (ii)  $B \in \Theta_X$  if and only if there exist a FRV  $A$  such that  $B \oplus A = X$ ,  $\mathbb{P}$ -a.s.. If  $X \in L^2[\Omega; \mathbb{F}]$ , then  $A$  is in  $L^2[\Omega; \mathbb{F}]$  too.
- (iii)  $\Theta_X$  is a convex subset in  $(\mathbb{F}, \oplus)$ . As a consequence, if  $B \in \Theta_X$ , then  $\{\lambda B\}_{\lambda \in [0,1]} \subseteq \Theta_X$ .
- (iv)  $\Theta_X$  is a closed subset of  $(\mathbb{F}, \delta_H^\infty)$ .
- (v)  $\Theta_X$  is a closed subset of  $(\mathbb{F}, d_2)$ .

**Proof.** Using the definition of  $\theta_{X(\omega)}$  and the characterization of element in  $\mathbb{F}$  via the support functions, we get the following chains of equalities.

$$\begin{aligned} E &= \{\omega \in \Omega : \mathbf{Ste}(B) = 0 \text{ and } \exists A_\omega \in \mathbb{F}, B \oplus A_\omega = X(\omega)\} \\ &= \{\omega \in \Omega : \mathbf{Ste}(B) = 0\} \cap \{\omega \in \Omega : \exists A_\omega \in \mathbb{F}, \text{ s.t. } s_B + s_{A_\omega} = s_{X(\omega)}\}. \end{aligned}$$

Since  $B$  is a deterministic fuzzy set,  $E_0 = \{\omega \in \Omega : \mathbf{Ste}(B) = 0\}$  is either the empty set or the whole  $\Omega$ ; hence  $E_0$  is measurable. On the other hand,  $A_\omega$  in  $\mathbb{F}$  satisfies  $B \oplus A_\omega = X(\omega)$  if and only if  $s_B + s_{A_\omega} = s_{X(\omega)}$  or, equivalently, if and only if  $s_{X(\omega)} - s_B$  is the support function of some element in  $\mathbb{F}$ . Thus, because of Properties 1–6 we have that

$$\begin{aligned} E &= E_0 \cap \{\omega \in \Omega : f_\omega \text{ satisfies Properties 1–6}\} \\ &= E_0 \cap E_1 \cap \dots \cap E_6, \end{aligned}$$

where  $E_i = \{\omega \in \Omega : f_\omega \text{ satisfies Property } i\}$  for  $i = 1, \dots, 6$ . If  $E_1, \dots, E_6$  are measurable events, then  $E$  is measurable too. To show this note that each  $E_i$  ( $i = 1, \dots, 6$ ) can be written as  $E_i = \{\omega : g_i(\omega) \leq 0\}$  where

$$\begin{aligned} g_1 &= \sup\{|\limsup_{y \rightarrow x} f_\omega(y) - f_\omega(x)| : x \in B(0, 1)\}, \\ g_2 &= \sup\{\lambda f_\omega(x) - f_\omega(\lambda x) : \lambda \in (0, 1], x \in B(0, 1)\}, \\ g_3 &= \sup\left\{\|x\|f_\omega\left(\lambda \frac{x}{\|x\|}\right) - \|x_1\|f_\omega\left(\lambda \frac{x_1}{\|x_1\|}\right) - \|x_2\|f_\omega\left(\lambda \frac{x_2}{\|x_2\|}\right) \right. \\ &\quad \left. : \lambda \in (0, 1], x, x_1, x_2 \in \mathbb{R}^d \setminus \{0\}, \text{ with } x = x_1 + x_2\right\}, \\ g_4 &= -\sup\{f_\omega(x) + f_\omega(-x) : x \in B(0, 1)\}, \\ g_5 &= \sup\left\{\frac{|f_\omega(x)|}{\|x\|} : x \in B(0, 1) \setminus \{0\}\right\}, \\ g_6 &= |f_\omega(0)|. \end{aligned}$$

Clearly  $\omega \mapsto g_i(\omega)$  are measurable maps and hence  $E$  is a measurable event in the  $\sigma$ -algebra  $\mathfrak{F}$ .

**ITEM (i).** Surely  $\mathbb{I}_0$  belongs to  $\Theta_X$ , hence  $\Theta_X$  is not empty.

**ITEM (ii).** The sufficiency is trivial, let us prove the necessity. Let  $E^c = \Omega \setminus E = \{\omega \in \Omega : B \notin \theta_{X(\omega)}\}$ , by hypothesis  $\mathbb{P}(E) = 1$  and  $\mathbb{P}(E^c) = 0$ . For every  $\omega \in \Omega \cap E$ , there exists  $A_\omega \in \mathbb{F}$  such that  $B \oplus A_\omega = X(\omega)$ . Let us consider the map

$$\begin{aligned} A : \quad \Omega &\rightarrow \mathbb{F} \\ \omega &\mapsto A(\omega) = \begin{cases} A_\omega, & \omega \in \Omega \cap E, \\ \mathbb{I}_0, & \omega \in E^c. \end{cases} \end{aligned} \tag{4}$$

Since  $s_A = s_X - s_B$   $\mathbb{P}$ -almost surely,  $s_A$  is measurable. Hence, the map  $A$  defined above, is the FRV we are looking for.

Moreover, let  $X \in L^2[\Omega; \mathbb{F}]$ , then  $s_X$  and hence  $s_A = s_X - s_B$  belong to  $L^2[\Omega; \mathcal{L}]$ .

**ITEM (iii).** Consider  $B_1, B_2 \in \Theta_X$ . From above part we know that there exist two FRV  $A_1, A_2$  with values in  $\mathbb{F}$  such that  $\mathbb{P}$ -a.s.  $B_1 \oplus A_1 = X$  and  $B_2 \oplus A_2 = X$ . For any  $\lambda \in [0, 1]$ , the following hold

$$\lambda(B_1 \oplus A_1) = \lambda X, \quad (1 - \lambda)(B_2 \oplus A_2) = (1 - \lambda)X, \quad \mathbb{P} - a.s.$$

from which we get

$$\lambda B_1 \oplus (1 - \lambda)B_2 \oplus A = X, \quad \mathbb{P} - a.s.$$

with  $A = \lambda A_1 \oplus (1 - \lambda)A_2$   $\mathbb{P}$ -a.s.. Hence  $\lambda B_1 \oplus (1 - \lambda)B_2 \in \Theta_X$ .

To prove the last part consider  $B \in \Theta_X$ , then  $\lambda B = \lambda B \oplus (1 - \lambda)\mathbb{I}_0 \in \Theta_X$ .

**ITEM (iv).** Consider a sequence  $\{B_n\}_{n=1}^\infty \subset \Theta_X$  converging to  $B \in \mathbb{F}$  in  $(\mathbb{F}, \delta_H^\infty)$ , i.e.

$$\delta_H^\infty(B, B_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

We have to prove that  $B \in \Theta_X$ . For any  $n \in \mathbb{N}$ , let  $E_n = \{\omega \in \Omega : B_n \in \Theta_X\}$  and  $A_n$  a FRV as in (ii). Then for every  $\omega \in \Omega \cap E_n$ ,  $B_n \oplus A_n(\omega) = X(\omega)$  and

$$\delta_H^\infty(A_n(\omega), A_n(\omega)) = \delta_H^\infty(B_n, B_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, the completeness of  $(\mathbb{F}, \delta_H^\infty)$  guarantees that, for every  $\omega \in \Omega \setminus \bigcup_n (E_n)^c = \Omega \cap \bigcap_n E_n$ ,  $\{A_n(\omega)\}_{n \in \mathbb{N}}$  converges w.r.t.  $\delta_H^\infty$  to some  $A_\omega \in \mathbb{F}$ . Further, for every  $\omega \in \Omega \cap \bigcap_n E_n$  and  $n \in \mathbb{N}$  the following inequalities hold

$$\begin{aligned} 0 \leq \delta_H^\infty(X(\omega), B \oplus A_\omega) &\leq \delta_H^\infty(X(\omega), B_n \oplus A_n(\omega)) + \delta_H^\infty(B_n \oplus A_n(\omega), B \oplus A_\omega) \\ &\leq 0 + \delta_H^\infty(B_n, B) + \delta_H^\infty(A_n(\omega), A_\omega) \rightarrow 0 \end{aligned}$$

where, for the first addend, we use the fact that  $X(\omega) = B_n \oplus A_n(\omega)$ . Then  $X = B \oplus A$   $\mathbb{P}$ -a.s., and  $A$  is the FRV defined by Equation (4). Thus we have the thesis; the limit of the convergent sequence  $\{B_n\} \subseteq \Theta_X$  belongs to  $\Theta_X$  too.

**ITEM (v).** Let us consider a sequence  $\{B_n\}_{n=1}^\infty \subset \Theta_X$  converging to  $B \in \mathbb{F}$  in  $(\mathbb{F}, d_2)$ , i.e.

$$d_2(B, B_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

We have to prove that  $B \in \Theta_X$ . In this case,  $(\mathbb{F}, d_2)$  is not complete and, hence, we can not repeat all arguments in (iv). In particular, for any  $n \in \mathbb{N}$  and for every  $\omega \in \Omega \cap E_n = \{\omega \in \Omega : B_n \in \theta_{X(\omega)}\}$ , there exist  $A_n(\omega)$  such that  $B_n \oplus A_n(\omega) = X(\omega)$  and, using analogous arguments of those in Proposition 1,

$$d_2(A_m(\omega), A_n(\omega)) = \left( \int_0^1 \int_{S^{d-1}} |s_{A_m(\omega)}(\alpha, u) - s_{A_n(\omega)}(\alpha, u)|^2 du d\alpha \right)^{\frac{1}{2}} = d_2(B_m, B_n) \rightarrow 0,$$

as  $n \rightarrow \infty$  and where  $d\alpha$  and  $du$  denote the Lebesgue measure on  $[0, 1]$  and the normalized Lebesgue measure on  $S^{d-1}$  respectively. Thus, for every  $\omega \in \Omega \cap \bigcap_n E_n$ ,  $\{s_{A_n(\omega)}\}_{n \in \mathbb{N}}$  is a Cauchy sequence in the Hilbert space  $\mathcal{L} (= L^2([0, 1] \times S^{d-1}; \mathbb{R}))$  and it admits limit in  $\mathcal{L}$ , namely  $f_\omega$ . Since

$$\|s_{A_n(\omega)} - (s_{X(\omega)} - s_B)\|_{\mathcal{L}} = \|(s_{A_n(\omega)} - s_{X(\omega)}) + s_B\|_{\mathcal{L}} = \|s_B - s_{B_n}\|_{\mathcal{L}} \rightarrow 0,$$

necessarily we have

$$s_{A_n(\omega)} \xrightarrow{L^2} f_\omega = s_{X(\omega)} - s_B, \quad \forall \omega \in \Omega \cap \bigcap_n E_n.$$

Note that,  $f_\omega$  is not necessarily the support function of some element in  $\mathbb{F}$ . In other words, for every  $\omega \in \Omega \cap \bigcap_n E_n$ ,  $\{A_n(\omega)\}_{n \in \mathbb{N}}$  is a Cauchy sequence in the non-complete space  $(\mathbb{F}, d_2)$ , but under the embedding  $j$ , Equation (2), we have that the sequence  $\{j(A_n(\omega))\}_{n \in \mathbb{N}} = \{s_{A_n(\omega)}\}_{n \in \mathbb{N}}$  is a Cauchy sequence that admits limit in the Hilbert space  $\mathcal{L}$ . But, in general, this limit is not the image under  $j$  of some element of  $\mathbb{F}$ . We claim that, for every  $\omega \in \Omega \cap \bigcap_n E_n$ , there exists  $A_\omega \in \mathbb{F}$  such that  $s_{A_\omega} = f_\omega = s_{X(\omega)} - s_B$ . This allows us to deduce the thesis because, defining the FRV  $A$  as in Equation (4), we have that  $B \oplus A = X$  holds  $\mathbb{P}$ -a.s..

In fact, let us consider the family  $\{C_\alpha\}_{\alpha \in [0, 1]}$  of subsets of  $\mathbb{R}^d$  defined by

$$C_\alpha = \{y \in \mathbb{R}^d : \langle y, u \rangle \leq f_\omega(\alpha, u), \forall u \in S^{d-1}\}, \quad \alpha \in [0, 1].$$

In what follows, let  $\omega \in \Omega \cap \bigcap_n E_n$ , we prove that the family  $\{C_\alpha\}_{\alpha \in [0, 1]}$  satisfies (a), (b), (c) from Lemma 2, and it defines uniquely a fuzzy set  $\nu$  whose support function is, clearly,  $f_\omega$ . Thus the fuzzy set  $\nu$  defined in Lemma 2 is just the  $A(\omega)$  in  $\mathbb{F}$  we are looking for.

(a). Let  $\alpha \in [0, 1]$ .

$C_\alpha$  is non-empty: since  $B_\alpha \subseteq (X(\omega))_\alpha$ , then for every  $u \in S^{d-1}$

$$f_\omega(\alpha, u) = s_{X(\omega)}(\alpha, u) - s_B(\alpha, u) \geq 0 = \langle 0, u \rangle, \quad (5)$$

i.e.  $0 \in C_\alpha$ .

$C_\alpha$  is convex: let  $\lambda \in [0, 1]$  and  $y_1, y_2 \in C_\alpha$ , for every  $u \in S^{d-1}$

$$\langle \lambda y_1 + (1 - \lambda)y_2, u \rangle \leq \lambda f_\omega(\alpha, u) + (1 - \lambda)f_\omega(\alpha, u) = f_\omega(\alpha, u)$$

i.e.  $\lambda y_1 + (1 - \lambda)y_2 \in C_\alpha$ .

$C_\alpha$  is compact: we have to prove that it is a bounded closed subset of  $\mathbb{R}^d$ . Note that  $\{0\} \subseteq B_\alpha \subseteq (X(\omega))_\alpha$ , then  $s_{X(\omega)}(\alpha, u) \geq s_B(\alpha, u) \geq 0$  for each  $u \in S^{d-1}$  and  $s_{X(\omega)}(\alpha, u) \geq s_{X(\omega)}(\alpha, u) - s_B(\alpha, u) = f_\omega(\alpha, u)$ .

This implies that  $\langle y, u \rangle$  is bounded for every  $u \in S^{d-1}$  and hence that  $C_\alpha \subseteq \mathbb{R}^d$  is bounded. On the other hand, let  $\{y_n\} \subset C_\alpha$  be convergent to  $y \in \mathbb{R}^d$ , then, for every  $n \in \mathbb{N}$  and  $u \in S^{d-1}$ ,

$$\langle y_n, u \rangle \leq f_\omega(\alpha, u),$$

and passing to the limit we obtain the same inequality for  $y$  and for every  $u \in S^{d-1}$ ; i.e.  $y \in C_\alpha$ . This fact allows us to conclude that  $C_\alpha$  is closed and hence compact.

(b). Let  $0 \leq \alpha \leq \beta \leq 1$ . Note that, for every  $n \in \mathbb{N}$  and  $u \in S^{d-1}$ ,  $s_{A_n(\omega)}(\beta, u) \leq s_{A_n(\omega)}(\alpha, u)$ . Let  $n \rightarrow \infty$ , then  $f_\omega(\beta, u) \leq f_\omega(\alpha, u)$  for every  $u \in S^{d-1}$ ; i.e., for every  $u \in S^{d-1}$  and  $n \in \mathbb{N}$ ,  $s_{A_n(\omega)}$  and  $f_\omega$  are non-increasing functions with respect to  $\alpha$ . Now, let us consider  $y \in C_\beta$ , then for every  $u \in S^{d-1}$ ,  $\langle y, u \rangle \leq f_\omega(\beta, u) \leq f_\omega(\alpha, u)$ ; i.e.  $y \in C_\alpha$  and  $C_\beta \subseteq C_\alpha$ .

(c). Let  $\{\alpha_i\}_{i \in \mathbb{N}} \subset [0, 1]$  such that  $\alpha_i \uparrow \alpha$  as  $i$  tends to infinity, that is  $\alpha_i \leq \alpha_{i+1}$  and  $\alpha_i \rightarrow \alpha$  as  $i \rightarrow \infty$ . Because of  $\alpha_i \leq \alpha$  and (b), we have  $C_\alpha \subseteq C_{\alpha_i}$  and  $C_\alpha \subseteq \bigcap_{i \in \mathbb{N}} C_{\alpha_i}$ . It remains to show the opposite inclusion. To do this let  $y \in \bigcap_{i \in \mathbb{N}} C_{\alpha_i}$ , i.e.  $y \in C_{\alpha_i}$  for all  $i \in \mathbb{N}$  or, equivalently,

$$\langle y, u \rangle \leq f_\omega(\alpha_i, u), \quad \text{for every } i \in \mathbb{N}, u \in S^{d-1}. \quad (6)$$

Note that, for every  $u \in S^{d-1}$ ,  $f_\omega(\cdot, u)$  is left-continuous with respect to  $\alpha$  because it is the difference of two left-continuous functions (cf. Equation (5)). Hence, for the arbitrariness of  $i$  in (6), as  $i$  tends to infinity we get  $\langle y, u \rangle \leq f_\omega(\alpha, u)$ ; i.e.  $y \in C_\alpha$ .  $\blacksquare$

### 3 Hukuhara decomposition

Let us recall again the well known decomposition (1) for Gaussian FRV  $X$

$$X = \mathbb{E}X \oplus \mathbb{I}_\xi,$$

where  $\mathbb{E}X$  is the Aumann expectation of  $X$ , and  $\xi$  is a Gaussian random element in  $\mathbb{R}^d$  with  $\mathbb{E}\xi = 0$ . Equation (1) implies a *randomness defuzzification* for FRV  $X$  that is equal to its expected value  $\mathbb{E}X$  (a deterministic fuzzy set) up to a random Gaussian translation  $\xi$ . In [2], the author showed another case of defuzzification of randomness: a Brownian fuzzy set-valued process is reduced to be a Brownian process in  $\mathbb{R}^d$ . In both cases, the randomness initially defined on  $\mathbb{F}$  can be simply defined on  $\mathbb{R}^d$ .

Now, our question becomes the following one. *Under what conditions can we establish that a defuzzification of randomness occurs for fuzzy set-valued random process?* In other words: *Can a fuzzy process, whose randomness is given only by vectors, be characterized in some way?* In this section we propose a positive answer to the above question. We focus mainly on a decomposition theorem for FRV. In fact, in Theorem 6, we prove that any FRV  $X$  can be decomposed as the sum of a deterministic convex fuzzy set  $H_X^\perp$  and a FRV  $Y$  (that contains the whole randomness) in a unique way. This decomposition allows us to characterize, by means of the Aumann expected value, the FRV that is a random translation of a deterministic fuzzy set.

**Definition 4** A FRV  $X$  is a *translation* if there exists  $M \in \mathbb{F}$  with  $\mathbf{Ste}(M) = 0$  such that

$$X(\omega) = M \oplus \mathbb{I}_{\mathbf{Ste}(X)}.$$

Roughly speaking, the randomness of a translation depends only on the specific location in the underlying space  $\mathbb{R}^d$  while it does not depend on its fuzzy shape. Note that, accordingly to (1), every Gaussian FRV  $X$  is a FRV translation with  $M \oplus \mathbb{I}_{\mathbb{E}(\mathbf{Ste}(X))} = \mathbb{E}X$ . Another sufficient condition for  $X$  to be a FRV translation is given by Proposition 5, while a necessary and sufficient condition is state in Theorem 8.

**Proposition 5** *Let  $X$  be a FRV such that  $\mathbb{E}X = \mathbb{I}_c$  where  $c \in \mathbb{R}^d$ . Then  $X = \mathbb{I}_\xi$   $\mathbb{P}$ -a.s. for some random element  $\xi$  in  $\mathbb{R}^d$ . (Clearly  $X$  is a FRV translation.)*

**Proof.** Thesis can be obtained using similar arguments in [2, Theorem 8], or, whenever  $X \in L^2[\Omega; \mathbb{F}]$ , as corollary of the Theorem 6 and Theorem 8.  $\blacksquare$

Clearly, the vice versa of Proposition 5 does not hold, for example in the case of Gaussian FRV. In order to characterize translation FRV, we need the following decomposition theorem.



**Theorem 6** *Let  $X \in L^2[\Omega; \mathbb{F}]$  with  $\mathbf{Ste}(X) = 0$ . Thus there exists  $H_X^\perp \in \mathbb{F}$  with  $\mathbf{Ste}(H_X^\perp) = 0$  and  $Y \in L^2[\Omega; \mathbb{F}]$  such that  $X$  decomposes according to*

$$X(\omega) = H_X^\perp \oplus Y(\omega), \quad (7)$$

*for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . In particular,  $H_X^\perp$  is the unique element in  $\mathbb{F}$  that satisfies (7) and minimizes  $\mathbb{E}[(d_2(X, C))^2]$ ; i.e., there exists a unique  $H_X^\perp \in \Theta_X$  such that*

$$H_X^\perp := \arg \min_{B \in \Theta_X} \mathbb{E}[(d_2(X, B))^2]. \quad (8)$$

*Hence  $Y$  is the unique (except on a  $\mathbb{P}$ -negligible set) FRV such that its support function is given by  $s_Y = s_X - s_{H_X^\perp}$ . Moreover,  $H_X^\perp$  is a maximal element in  $\Theta_X$  with respect to the level-wise set inclusion; that is, if  $C \in \Theta_X$  with  $(H_X^\perp)_\alpha \subseteq C_\alpha$  for any  $\alpha \in [0, 1]$ , then  $H_X^\perp = C$ .*

**Proof.** Since  $\Theta_X$  collects all the element of  $\mathbb{F}$  for which (7) holds, we have to prove that there exists a unique element in  $\Theta_X$  that minimizes the map  $B \in \Theta_X \rightarrow \mathbb{E}[(d_2(X, B))^2]$ . At first note that  $\Theta_X$  can be seen as a subset of  $L^2[\Omega; \mathbb{F}]$ ; in fact, for each  $B \in \Theta_X$  the constant map  $\omega \mapsto B$  is an element of  $L^2[\Omega; \mathbb{F}]$  since

$$\mathbb{E}[(\sup_{b \in B_0} \|b\|)^2] = (\sup_{b \in B_0} \|b\|)^2 < +\infty.$$

Moreover,  $\Theta_X$  is closed in  $L^2[\Omega; \mathbb{F}]$  as a consequence of

$$\mathbb{E}[(d_2(A, B))^2] = (d_2(A, B))^2,$$

for any couples  $A, B \in \mathbb{F}$ , and thanks to the fact that  $\Theta_X$  is closed in  $(\mathbb{F}, d_2)$ , see Proposition 3.

Thus the minimization problem is equivalent to prove that there exists a unique projection of  $X$  onto  $\Theta_X$  that is a closed convex subset of  $L^2[\Omega; \mathbb{F}]$  endowed with the metric  $\Delta_2$ . Since  $L^2[\Omega; \mathbb{F}]$  embeds isometrically in the Hilbert space  $L^2[\Omega; \mathcal{L}]$  through map  $J$  (see the Introduction), there exists a unique element  $H_X^\perp \in \Theta_X$  that realizes the required minimum (8).

As a consequence of  $H_X^\perp \in \Theta_X$  and of (ii) in Proposition 3, the FRV  $Y$  is defined through its support function  $s_Y = s_X - s_{H_X^\perp}$ .

Finally, let  $C$  be as in the thesis; thus inclusions  $(H_X^\perp)_\alpha \subseteq C_\alpha \subseteq X_\alpha$  imply  $s_X - s_C \leq s_X - s_{H_X^\perp}$ . Then, by definition of  $H_X^\perp$  and  $d_2$ , necessarily  $C = H_X^\perp$  holds.  $\blacksquare$

The chosen notation wants to recall the line of the proof;  $H_X^\perp$  is obtained through a projection theorem of the given FRV  $X$  on its Hukuhara set  $\Theta_X$ . Further, we want to stress out that the suffix  $X$  does not mean that  $H_X^\perp$  is random; in fact, it does not depend on  $\omega$  but rather it is a deterministic element of  $\mathbb{F}$  (that is a constant element in  $L^2[\Omega; \mathbb{F}]$ ) that depends on the whole map  $\omega \mapsto X(\omega)$ .

The following theorems provide necessary and sufficient condition for a FRV to be a translation.

**Theorem 7** *Let  $X$  be a FRV translation, and  $\tilde{X} = X \oplus \mathbb{I}_{-\mathbf{Ste}(X)}$ . Then*

$$X = H_X^\perp \oplus \mathbb{I}_{\mathbf{Ste}(X)}, \quad \mathbb{P} - a.s. \quad (9)$$

**Proof.** By hypothesis  $X = M \oplus \mathbb{I}_{\mathbf{Ste}(X)}$  for some  $M \in \mathbb{F}$  with  $\mathbf{Ste}(M) = 0$ . Clearly,  $\tilde{X} = X \oplus \mathbb{I}_{-\mathbf{Ste}(X)} = M$  and  $\mathbf{Ste}(\tilde{X}) = 0$ . Thus, by Theorem 6 applied to  $\tilde{X}$ , we have  $M \in \Theta_{\tilde{X}}$  and  $\mathbb{E}[(d_2(M, \tilde{X}))^2] = 0$ ; that is,  $M = H_{\tilde{X}}^\perp$ .  $\blacksquare$

**Theorem 8** *Let  $X \in L^2[\Omega; \mathbb{F}]$ .  $X$  is a FRV translation if and only if  $H_X^\perp$  satisfies*

$$\mathbb{E}X = H_X^\perp \oplus \mathbb{I}_{\mathbb{E}(\mathbf{Ste}(X))} \quad (10)$$

*with  $\mathbb{E}X$  being the Aumann expectation; in other words,  $H_X^\perp$  is  $\mathbb{E}X$  up to a translation.*

**Proof.** For the “only if” part, in order to obtain Equation (10), it is sufficient to compute the expectation in Equation (9).

Consider the “if” part. For the sake of simplicity, let us assume that  $\mathbf{Ste}(X) = 0$ , a straightforward

argument extends the result in the more general case of a FRV with non-void  $\mathbf{Ste}(X)$ . Then, in term of support functions, Equation (7) becomes

$$s_X = s_{H_X^\perp} + s_Y = s_{\mathbb{E}X} + s_Y, \quad \mathbb{P} - a.s.$$

where we use the fact that  $H_X^\perp = \mathbb{E}X$ . Computing expectation of both sides and using (3), we get  $s_{\mathbb{E}Y} = 0$ . Hence  $Y = \mathbb{I}_\xi$  a.s. for some random element  $\xi$  in  $\mathbb{R}^d$  (cf. [2]).  $\blacksquare$

**Remark 9** Whenever  $X \in L^2[\Omega; \mathbb{F}]$ , in view of Theorem 6 and Theorem 8, we get a proof of Proposition 5. In fact, suppose that  $\mathbb{E}X = \mathbb{I}_c$  for some  $c \in \mathbb{R}^d$ , and compute expectation of both sides in Equation (7)

$$\mathbb{I}_c = \mathbb{E}X = H_X^\perp \oplus \mathbb{E}Y.$$

Hence, for any  $\alpha \in [0, 1]$ ,  $(H_X^\perp)_\alpha$  is a subset of  $\{c\}$  up to a translation, that is  $(H_X^\perp)_\alpha$  is a singleton as well as  $(\mathbb{E}Y)_\alpha$ . Then  $H_X^\perp = \mathbb{I}_{c'}$  for some  $c' \in \mathbb{R}^d$ , i.e.  $H_X^\perp$  is equal to  $\mathbb{E}X$  up to a translation and, by Theorem 8,  $X$  is a FRV translation that implies  $Y = \mathbb{I}_\xi$  for some random element in  $\mathbb{R}^d$ . Finally, Equation (7) becomes

$$X = H_X^\perp \oplus Y = \mathbb{I}_{c'} \oplus \mathbb{I}_\xi = \mathbb{I}_{\xi'},$$

that is the thesis of Proposition 5.  $\square$

Moreover, the following results hold.

**Corollary 10** *Let  $X \in L^2[\Omega; \mathbb{F}]$  with  $\mathbf{Ste}(X) = 0$  and  $\mathbb{E}X = H_X^\perp$ . Thus  $X$  is almost surely deterministic and equal to  $H_X^\perp$ .*

**Corollary 11** *Let  $X \in L^2[\Omega; \mathbb{F}]$ ,  $D \in \mathbb{F}$  and  $X' = X \oplus D$  with  $\mathbf{Ste}(X) = \mathbf{Ste}(D) = 0$  (hence  $\mathbf{Ste}(X') = 0$  too). Then  $H_{X'}^\perp = H_X^\perp \oplus D$ .*

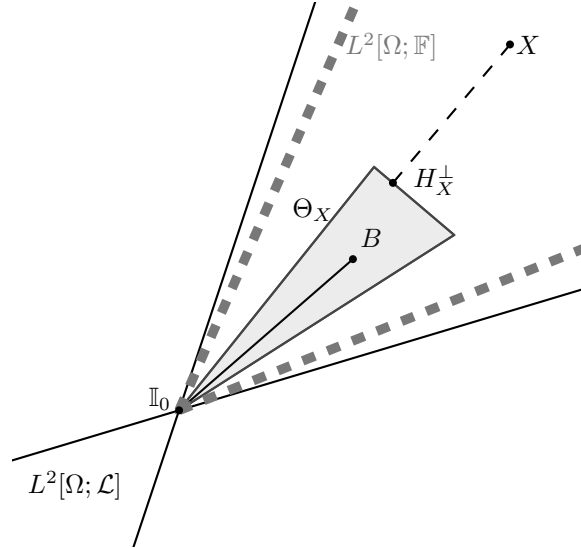


Figure 1: A qualitative graphical interpretation of some results of Section 2 and Section 3. In particular,  $\Theta_X$  is represented as a closed convex subset of  $\mathbb{F}$  containing the origin and such that, for any  $B \in \Theta_X$  and  $\lambda \in [0, 1]$ ,  $\lambda B \in \Theta_X$ . Hence,  $H_X^\perp$  is the projection of  $X$  on  $\Theta_X$ , as a subset of  $L^2[\Omega; \mathbb{F}]$ , with respect to the metric  $\mathbb{E}[d_2(\cdot, \cdot)^2]$ , this also guarantees the uniqueness of  $H_X^\perp$  since the cone  $L^2[\Omega; \mathbb{F}]$  is embeddable in the Hilbert space  $L^2[\Omega; \mathcal{L}]$  through the isometry  $X \mapsto j(X)$ . Finally the following inclusions or embeddings are qualitatively represented:  $\Theta_X \subseteq \mathbb{F} \hookrightarrow L^2[\Omega; \mathbb{F}] \hookrightarrow L^2[\Omega; \mathcal{L}]$ .

Remark 12 shows an example of an  $X$  in  $L^2[\Omega; \mathbb{F}]$  with  $\mathbf{Ste}(X) = 0$  for which  $\mathbb{E}(X) \neq H_X^\perp$  and for which  $H_X^\perp$  is not necessarily  $\mathbb{I}_0$ ; i.e., in terms of Theorem 8,  $X$  is not a translation but its deterministic part  $H_X^\perp$  in the decomposition (7) is not just reduced to the origin.

**Remark 12** Let  $\mathbb{R}^d = \mathbb{R}$ ,  $(\Omega = [0, 1], \mathcal{B}_{[0,1]}, \mathbb{P})$  where  $\mathcal{B}_{[0,1]}$  denotes the Borel  $\sigma$ -algebra on  $[0, 1]$  w.r.t. the euclidean metric and  $\mathbb{P} = \mu$  is the Lebesgue measure. Let  $X$  be the FRV defined by  $X := \mathbb{I}_{[\omega, \omega]}$ , for any  $\omega \in [0, 1]$ . Clearly  $X \in L^2[\Omega; \mathbb{F}]$  and  $\mathbf{Ste}(X) = 0$ . Moreover,

$$f_m(\omega) := \min X_1(\omega) = -\omega \quad \text{and} \quad f_M(\omega) := \max X_1(\omega) = \omega$$

are integrable selections of the 1-level RaCS  $X_1$ . Obviously, any other integrable selection  $f$  of  $X_1$  satisfies

$$f_m(\omega) \leq f(\omega) \leq f_M(\omega), \quad \text{for each } \omega \in [0, 1].$$

Then

$$-\frac{1}{2} = \mathbb{E}f_m \leq \mathbb{E}f \leq \mathbb{E}f_M = \frac{1}{2},$$

and, by the convexity of Aumann expectation and because  $X_1 = X_\alpha$  for any  $\alpha \in [0, 1]$ ,  $\mathbb{E}X_1 = [-\frac{1}{2}, \frac{1}{2}] = \mathbb{E}X_\alpha$ , that is  $\mathbb{E}X = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}$ .

We prove that  $\mathbb{E}X \notin \Theta_X$  and hence, by Theorem 10,  $X$  is not a FRV translation. In fact, note that

$$X \ominus_H \mathbb{E}X = \mathbb{I}_{[-\omega, \omega]} \ominus_H \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]} = \begin{cases} \mathbb{I}_{[-\omega+\frac{1}{2}, \omega-\frac{1}{2}]}, & \omega > \frac{1}{2}, \\ \mathbb{I}_0, & \omega = \frac{1}{2}, \\ \text{it does not exist}, & \omega < \frac{1}{2}, \end{cases}$$

implies

$$\mathbb{P}(\mathbb{E}X \in \theta_X) = \mathbb{P}(\text{there exists } X \ominus_H \mathbb{E}X) = \mathbb{P}\left(\omega \geq \frac{1}{2}\right) = \frac{1}{2},$$

and hence  $\mathbb{E}X \notin \Theta_X$ .

Actually we can show that  $\Theta_X = \{\mathbb{I}_0\}$  and hence  $H_X^\perp = \mathbb{I}_0$ . In fact, by absurd let  $B \in \Theta_X$  with  $B \neq \mathbb{I}_0$ , then there exists  $\alpha \in [0, 1]$  such that  $B_\alpha = [a, b]$  with  $a < b$  and there exists  $X_\alpha \ominus_H B_\alpha$ , here  $\ominus_H$  is considered as the Hukuhara difference for subsets in  $\mathbb{R}$ . On the other hand

$$[-\omega, \omega] \ominus_H [a, b] = \begin{cases} [-\omega - a, \omega - b], & \omega - b > -\omega - a, \\ \{-\frac{b+a}{2}\}, & \omega = \frac{b-a}{2}, \\ \text{it does not exist}, & \omega < \frac{b-a}{2}, \end{cases}$$

and, as consequence,

$$\mathbb{P}([- \omega, \omega] \ominus_H [a, b] \text{ does not exist}) = \mu\left[\left(-\infty, \frac{b-a}{2}\right) \cap [0, 1]\right] > 0$$

where the last inequality is due to the fact that, by hypothesis,  $b - a > 0$ . This is an absurd since  $B \in \Theta_X$  by hypothesis. Thus  $\Theta_X = \{\mathbb{I}_0\} \neq \mathbb{E}X = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}$ .

Finally, in order to produce a more general example, let us consider

$$X = \mathbb{I}_{[-\omega, \omega]} \oplus \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]} = \mathbb{I}_{[-\omega-\frac{1}{2}, \omega+\frac{1}{2}]}$$

so that, from Corollary 11, we immediately obtain that

$$\mathbb{I}_{[-1, 1]} = \mathbb{E}X \neq H_X^\perp = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}.$$

Note that, this is a case in which  $H_X^\perp$  is different from  $\mathbb{I}_0$ . □

## 4 Conclusion

In this paper, we have proven that any square integrable FRV can be decomposed as  $X = H_X^\perp \oplus Y$ , where  $H_X^\perp$  is a unique deterministic fuzzy convex compact set (i.e. in  $\mathbb{F}$ ) and  $Y$  is an element of  $L^2[\Omega; \mathbb{F}]$ . This decomposition leads us to characterize FRV translations for which  $H_X^\perp = \mathbb{E}X$ , where the expectation is in the Aumann sense.

This fact is important, for example, in view of Proposition 5 that allows us to defuzzificate the randomness of a FRV process  $\{X_t\}_{t \geq 0}$  for which  $\mathbb{E}X_t = \mathbb{I}_c$  holds at any time  $t$ . In fact, since  $X_t$  is a translation at each  $t$ , it can be interpreted simply as a random element on  $\mathbb{R}^d$ .

In general, working with a centered  $X$  in  $L^2[\Omega; \mathbb{F}]$  one may distinguish different cases:

- the case of defuzzificated randomness, for which  $\mathbb{E}X = H_X^\perp$  and hence  $X$  is a translation.
- the case for which  $\mathbb{E}X \notin \Theta_X$  and  $H_X^\perp = \mathbb{I}_0$ ; the randomness of  $X$  is totally fuzzy.
- the case for which  $\mathbb{E}X \notin \Theta_X$  and  $H_X^\perp \neq \mathbb{I}_0$ . In this case, one may take advantages from decomposition  $X = H_X^\perp \oplus Y$  splitting the deterministic case from the random one.

Our decomposition of  $H_X^\perp$  is a particular case where the problem posed in [6, p.174–175] is well-solved by defining the Hukuhara set  $\Theta_X$ . In this view,  $H_X^\perp$  may be interpreted as an expectation for  $X$  that satisfies some of the properties, listed in [6, p.190] for random closed sets but trivially extendible in the fuzzy case, of a “reasonable” expectation of  $X$ .

The decomposition theorem proposed in Section 3 could not be compared with the fuzzy regression problem stated in [11]. In fact, in that paper, the authors look for the best linear approximation function of a given square integrable FRV  $Y$  by another square integrable FRV  $X$ , studying the minimization problem

$$\inf_{a \in \mathbb{R}, B \in \mathbb{F}} \mathbb{E}[d_2(Y, aX \oplus B)^2].$$

Future works may consider the possibility to relax some hypothesis; for example, replacing  $\mathbb{R}^d$  with an Hilbert or a Banach space (problems may arise considering the embedding  $j$  and hence the closure of the Hukuhara set  $\Theta_X$ ), or dropping convexity hypothesis and hence stating a decomposition theorem for a fuzzy random element whose level sets are not necessarily convex. Finally, note that we restricted our studies to the existence of a such  $H_X^\perp$ ; however, it is certainly interesting to establish whenever  $H_X^\perp$  could be explicitly computed, though even in particular cases.

## Acknowledgements

The authors would like to thank Prof. V. Capasso for the helpful, fruitful and long discussions during the preparation of this paper.

## References

- [1] V. N. Bobylev. Support function of a fuzzy set and its characteristic properties. *Mathematical Notes*, 37:281–285, 1985.
- [2] E. G. Bongiorno. A note on Fuzzy set-valued Brownian Motion. *arXiv:1109.6167*.
- [3] A. Colubi, J. S. Dominguez-Menchero, M. Lopez-Diaz, D. A. Ralescu. A  $D_E[0, 1]$  representation of random upper semicontinuous functions. *Proceedings of the American Mathematical Society*, 130, 3237–3242, 2002.
- [4] P. Diamond and P. Kloeden. *Metric spaces of fuzzy sets: theory and applications*. Wiley, 1994.
- [5] S. Li, Y. Ogura, and V. Kreinovich. *Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables*. Kluwer Academic Publishers Group, Dordrecht, 2002.
- [6] I. Molchanov. *Theory of random sets*. Springer. (2005)
- [7] W. Näther. Linear Statistical Inference for Random Fuzzy Data. *Statistics*, 29:3, 221–240, 1997.
- [8] W. Näther. On random fuzzy variables of second order and their application to linear statistical inference with fuzzy data. *Metrika*, 51, 201–221, 2000.
- [9] A. B. Ramos-Guajardo, A. Colubi, G. González-Rodríguez, M. A. Gil. One-sample tests for a generalized Fréchet variance of a fuzzy random variable. *Metrika*, 71, 185–202, 2010.
- [10] T. Vetterlein and M. Navara. Defuzzification using Steiner points. *Fuzzy Sets and Systems*, 157:1455–1462, 2006.
- [11] A. Wünsche and W. Näther. Least-squares fuzzy regression with fuzzy random variables. *Fuzzy Sets and Systems*, 130, 43–50, 2002.